Model-based Defeasible Reasoning

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Abstract. Well-known forms of KLM-style defeasible entailment can be defined syntactically, via formula-based manipulations, and semantically, using ranked models. While entailment algorithms based on such syntactic characterisations have been developed, algorithms that directly manipulate the underlying models have not been explored. We present and analyse several algorithms, based on ranked model semantics, for computing two prominent forms of defeasible entailment: *rational closure* and *lexicographic closure*. In each case, we define an abstract representation of the ranked model, an algorithm for its construction, and a suitable adaptation of existing entailment algorithms, compatible with the representation. We also clarify the distinction between two forms of lexicographic closure in the literature.

Keywords: knowledge representation and reasoning · defeasible reasoning · rational closure · lexicographic closure

1 Introduction

Knowledge representation and reasoning (KRR) is a subfield of artificial intelligence that attempts to formalise the expression of information and philosophical patterns of reasoning. Knowledge is encoded symbolically and collated in a structure referred to as a *knowledge base*. Reasoning services are then defined to facilitate drawing reasonable conclusions from such knowledge bases.

A simple, yet expressive logic-based approach to KRR is defined in *classical propositional logic* (or *propositional logic*).

While exhibiting many desirable characteristics, propositional logic has two fundamental limitations in its ability to mimic human reasoning.

Propositional logic cannot explicitly express *typicality* whereby certain implications usually hold but may have exceptions. It is also *monotonic*, meaning conclusions drawn from some knowledge base cannot be retracted with the addition of new knowledge [9]. Such retractions are crucial in formalising the idea that new knowledge may require a re-examination of past conclusions.

To address these shortcomings, defeasible approaches to reasoning have been proposed as nonmonotonic alternatives to classical forms of entailment. Unlike classical entailment, there is no obvious way defeasible entailment ought to behave.

Kraus, Lehmann and Magidor (KLM) [9] proposed a set of properties as a thesis for how to define a 'sensible' or 'rational' notion of defeasible entailment. The seminal KLM paper [9] set out to characterise the preferential model-theoretic approach to nonmonotonic entailment taken by Shoham in proof-theoretic terms applied to *consequence relations*, inspired by the work of Gabbay in [6].

Two such examples, which will be our primary focus in this paper, are *rational closure* [12] and *lexicographic closure* [11], each representing distinct, valid patterns of human reasoning.

In both cases, computing entailment for a given knowledge base has been defined based on semantics involving the ranking of formulas in the knowledge base [12]. Giordano et. al [7] provide an alternative but equivalent semantic characterisation of rational closure based on a form of defeasible entailment known as *minimal ranked entailment*. Casini et. al [4] extend this characterisation to lexicographic closure, a refinement of rational closure, noting that it too can be characterised by a specific ranked model.

This paper will focus on constructing model-based representations of these forms of entailment, algorithmically. We also show that the lexicographic ordering defined by Casini et al. [4] differs from the usual ordering defined by Lehmann [11].

2 Background

2.1 Propositional Logic

Language and Semantics We define a set $\mathcal{P}$ containing all atomic propositions, representing the most basic units of knowledge [1]. Formulas can consist of a single atom, the negations ($\neg$) of other formulas, or the combination of two other formulas using one of the binary connectives $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$. The set of all possible formulas is often referred to as $\mathcal{L}$ (the language of propositional logic). An interpretation is a function $\mathcal{I} : \mathcal{P} \rightarrow \{T, F\}$ which assigns truth values to each propositional atom. We denote the set of all propositional interpretations with $\mathscr{U}$. We say that an interpretation $\mathcal{I} \in \mathscr{U}$ satisfies a formula $\alpha \in \mathcal{L}$, denoted $\mathcal{I} \Vdash \alpha$, if α evaluates to true using the usual truth-functional semantics. We refer to a finite set of formulas as a knowledge base. We say that an interpretation $\mathcal{I}$ satisfies a knowledge base $\mathcal{K}$ if $\forall \alpha \in \mathcal{K}$, $\mathcal{I} \Vdash \alpha$. Interpretations that satisfy a knowledge base are referred to as models of that knowledge base. We use the notation $Mod(\mathcal{K})$ (or $[\![\mathcal{K}]\!]$) to refer to the set of models of a knowledge base $\mathcal{K}$ (similarly for a single formula).

Entailment Using the above model-based semantics, entailment (or logical consequence), denoted using the $\models$ symbol, can be defined. A knowledge base $\mathcal{K}$ entails a formula α, written as $\mathcal{K} \models \alpha$, if and only if $Mod(\mathcal{K}) \subseteq Mod(\alpha)$. Intuitively, whenever all the formulas in $\mathcal{K}$ are true under a given interpretation, such will be the case for α and so we are able to conclude α whenever we have $\mathcal{K}$.

2.2 Defeasible Reasoning

2.3 The KLM Framework and Extensions

Initially, KLM [9] extended propositional logic by defining a consequence relation $\mid\!\sim$ representing defeasible implications in an attempt to reasonably represent *typicality*. Extensions of this framework instead define $\mid\!\sim$ as an additional connective (where $\alpha \mid\!\sim \beta$, with propositional formulas α, β, is read as 'typically, if α, then β' [4]). This extended language is defined as $\mathcal{L}_P \coloneqq \mathcal{L} \cup \{\alpha \mid\!\sim \beta \mid \alpha, \beta \in \mathcal{L}\}$ [8]. The semantics of $\mid\!\sim$ are then defined using *ranked interpretations* [12].

Definition 1. *A ranked interpretation is a function $\mathcal{R} : \mathscr{U} \mapsto \mathcal{N} \cup \{\infty\}$, such that for every $i \in \mathcal{N}$, if there exists a $u \in \mathscr{U}$ such that $\mathcal{R}(u) = i$, then there must be a $v \in \mathscr{U}$ such that $\mathcal{R}(v) = j$ with $0 \leq j < i$, where $\mathscr{U}$ is the set of all possible propositional interpretations [8].*

Ranked interpretations, therefore, assign to each propositional interpretation, a rank (with lower ranks corresponding, semantically, with more typical interpretations and higher ranks with less typical 'worlds'). Worlds with a rank of ∞, according to the ranked interpretation, are impossible, whereas worlds with finite ranks are possible.

Satisfaction Given that ranked interpretations indicate the relative typicality of worlds, it makes sense to define whether a ranked interpretation satisfies a defeasible implication based on the most typical worlds in that interpretation. In order to define the 'most typical worlds', a definition of minimal worlds concerning a formula in $\mathcal{L}$ is required.

Definition 2. *Given a ranked interpretation $\mathcal{R}$ and any formula $\alpha \in \mathcal{L}$, it holds that $u \in [\![\alpha]\!]^{\mathcal{R}}$ (the models of α in $\mathcal{R}$) is minimal if and only if there is no $v \in [\![\alpha]\!]^{\mathcal{R}}$ such that $\mathcal{R}(v) < \mathcal{R}(u)$ [8].*

This defines the concept of the 'best α worlds' (i.e. the lowest ranked, or most typical, of the worlds in which α is true).

Definition 3. *Given a ranked interpretation $\mathcal{R}$ and a defeasible implication $\alpha \mid\!\sim \beta$, $\mathcal{R}$ satisfies $\alpha \mid\!\sim \beta$, written $\mathcal{R} \Vdash \alpha \mid\!\sim \beta$ if and only if for every s minimal in $[\![\alpha]\!]^{\mathcal{R}}$, $s \Vdash \beta$. If $\mathcal{R} \Vdash \alpha \mid\!\sim \beta$ then $\mathcal{R}$ is said to be a model of $\alpha \mid\!\sim \beta$ [8].*

Therefore, in order for a ranked interpretation $\mathcal{R}$ to satisfy a defeasible implication $\alpha \mid\!\sim \beta$, it need only satisfy $\alpha \rightarrow \beta$ in the most typical (lowest ranked) α worlds of $\mathcal{R}$.

In the case of a propositional formula $\alpha \in \mathcal{L}$, it is required that every finitely-ranked world in $\mathcal{R}$ satisfies α for $\mathcal{R}$ to satisfy α. This is consistent with the idea that propositional formulas, which do not permit exceptionality, should be satisfied in every plausible world of a ranked interpretation, if such a ranking is to satisfy the formula.

It is now possible to model knowledge that expresses typicality and thus handles exceptional cases more reasonably.

We refer to a finite set of defeasible implications as a defeasible knowledge base. Note that we can express any classical propositional formula $\alpha \in \mathcal{L}$ using the defeasible representation $\neg\alpha \mid\!\sim \bot$. Henceforth, we assume that knowledge bases are defeasible unless specified otherwise. We define the *materialisation* of a defeasible knowledge base, $\mathcal{K}$, as $\overrightarrow{\mathcal{K}} := \{\alpha \to \beta \mid \alpha \mid\!\sim \beta \in \mathcal{K}\}$ [4].

Entailment We seek reasonable forms of non-monotonic entailment that permit the retraction of conclusions in cases where added knowledge contradicts these conclusions. A set of postulates defines such entailment relations [9], which is extended to define more specific classes of entailment [12, 4]. We will look at two particular patterns of entailment, namely *rational closure* and *lexicographic closure* with a specific emphasis on their model-based semantics for computing entailment.

2.4 Rational Closure

Rational closure represents a prototypical pattern of defeasible reasoning (one that is highly conservative in abnormal cases) in the KLM framework. Lehmann and Magidor [12] propose that any other reasonable form of entailment, while possibly being more 'adventurous' in its conclusions, should endorse at least those assertions in the rational closure of the corresponding knowledge base.

There are two principal ways to compute the rational closure of a given knowledge base. The first is *minimal ranked entailment.* This approach defines rational closure and the semantics of the associated entailment relation using a unique ranked model for a given knowledge base. The second is an algorithmic approach involving ranking statements in the knowledge base [12].

Base Rank and Rational Closure Although the original focus of this paper was on the semantics of rational closure and its model-theoretic construction, it is necessary to address its syntactic/algorithmic characterisation as the two are closely related.

Casini et al. [4] provide an aforementioned algorithmic description of rational closure for computing entailment queries in terms of two sub-algorithms included as Algorithms 1 and 2. Algorithm 1 ranks the formulas of the knowledge base according to how exceptional their antecedents are, and Algorithm 2 then answers a given entailment query using the information provided by Algorithm 1.

Algorithm 1 `BaseRank`

Input: A knowledge base $\mathcal{K}$
Output: An ordered tuple $(R_0, ..., R_{n-1}, R_\infty, n)$
i := 0;
$E_0 := \overrightarrow{\mathcal{K}}$;
repeat
  $E_{i+1} := \{\alpha \rightarrow \beta \in E_i \mid E_i \models \neg\alpha\}$;
  $R_i := E_i \setminus E_{i+1}$;
  $i := i + 1$;
until $E_{i-1} \neq E_i$
$R_\infty := E_{i-1}$;
$n := i - 1$;
return $(R_0, ..., R_{n-1}, R_\infty, n)$;

Algorithm 2 `RationalClosure`

Input: A knowledge base $\mathcal{K}$, and a defeasible implication $\alpha \mid\!\sim \beta$
Output: **true**, if $\mathcal{K} \models\!\!\approx \alpha \mid\!\sim \beta$, and **false** otherwise
$(R_0, ..., R_{n-1}, R_\infty, n) :=$ BaseRank($\mathcal{K}$);
$i := 0$
$R := \bigcup_{i=0}^{j<n} R_j$;
while $R_\infty \cup R \models \neg\alpha$ and $R \neq \emptyset$ **do**
  $R := R \setminus R_i$;
  $i := i + 1$;
end while
return $R_\infty \cup R \models \alpha \rightarrow \beta$;

Minimal Ranked Entailment A partial order over all ranked models of a knowledge base $\mathcal{K}$, denoted $\preceq_{\mathcal{K}}$, is defined as follows [4]:

Definition 4. *Given a knowledge base, $\mathcal{K}$, and $\mathcal{R}^{\mathcal{K}}$ the set of all ranked models of $\mathcal{K}$ (those ranked interpretations which satisfy $\mathcal{K}$), it holds for every $\mathcal{R}_1^{\mathcal{K}}, \mathcal{R}_2^{\mathcal{K}} \in \mathcal{R}^{\mathcal{K}}$ that $\mathcal{R}_1^{\mathcal{K}} \preceq_{\mathcal{K}} \mathcal{R}_2^{\mathcal{K}}$ if and only if for every $u \in \mathscr{U}$, $\mathcal{R}_1^{\mathcal{K}}(u) \leq \mathcal{R}_2^{\mathcal{K}}(u)$.*

Intuitively, this partial order favours ranked models that have their worlds 'pushed down' as far as possible [8]. It has a unique minimal element, $\mathcal{R}_{RC}^{\mathcal{K}}$, as shown by Giordano et al. [7]. We now define minimal ranked entailment using this minimal element as follows:

Definition 5. *Given a defeasible knowledge base $\mathcal{K}$, the minimal ranked interpretation satisfying $\mathcal{K}$, $\mathcal{R}_{RC}^{\mathcal{K}}$, defines an entailment relation, $\models\!\!\approx$, called minimal ranked entailment, such that for any defeasible implication $\alpha \mid\!\sim \beta$, $\mathcal{K} \models\!\!\approx \alpha \mid\!\sim \beta$ if and only if $\mathcal{R}_{RC}^{\mathcal{K}} \Vdash \alpha \mid\!\sim \beta$ [8].*

2.5 Lexicographic Closure

Lexicographic closure is a formalism of the presumptive pattern of reasoning introduced by Reiter [13] in the context of default logics. Presumptive reasoning is more 'adventurous' and willing to conclude statements so long as there is no evidence to the contrary (even in atypical cases). The semantics of lexicographic closure depends on a 'seriousness' ordering defined based on two criteria: specificity and cardinality.

Like rational closure, there are syntactic (formula-based) [11] and semantic (model-based) [4] descriptions of lexicographic closure.

Lehmann first defined lexicographic closure using a partial ordering on valuations [11]. This ordering favoured valuations with lower violation tuples, according to the natural lexicographic ordering of tuples. A violation tuple of a valuation is derived from the subset of a given defeasible knowledge base containing all the formulas the valuation violates. The tuple records the counts of formulas violated by the valuation ordered by seriousness (in this case, the base rank of the formula).

A formula-based algorithm for computing lexicographic closure, based on Lehmann's definition [11], successively produces weakened formula representations of each base rank. We refer to this algorithm as the `LexicographicClosure` algorithm [5], defined in Algorithm 3. It proceeds in the same manner as the `RationalClosure` algorithm but weakens each rank by considering incrementally smaller subsets of the rank instead of completely discarding the entire rank at each iteration.

Algorithm 3 `LexicographicClosure`

Input: A knowledge base $\mathcal{K}$, and a defeasible implication $\alpha \mid\!\sim \beta$
Output: **true**, if $\mathcal{K} \mid\!\approx_{LC} \alpha \mid\!\sim \beta$, and **false** otherwise
$(R_0, ..., R_{n-1}, R_\infty, n) :=$ BaseRank($\mathcal{K}$);
$i := 0$
$R := \bigcup_{i=0}^{j<n} R_j$;
while $R_\infty \cup R \models \neg\alpha$ and $R \neq \emptyset$ **do**
  $R := R \setminus R_i$;
  $m := \#R_i$ - 1;
  $R_{i,m} := \bigvee_{S \in \{T \subseteq R_i \mid \#T = m\}} \bigwedge_{s \in S} s$;
  while $R_\infty \cup R \cup \{R_{i,m}\} \models \neg\alpha$ and $m > 0$ **do**
    $m := m - 1$;
    $R_{i,m} := \bigvee_{S \in \{T \subseteq R_i \mid \#T = m\}} \bigwedge_{s \in S} s$
  end while
  $R := R \cup \{R_{i,m}\}$;
  $i := i + 1$;
end while
return $R_\infty \cup R \models \alpha \rightarrow \beta$;

Casini et al. provide another model-based definition of lexicographic closure in their framework of rational defeasible entailment relations [4]:

Definition 6. *$m \prec_{LC}^{\mathcal{K}} n$ if and only if $\mathcal{R}_{RC}^{\mathcal{K}}(n) = \infty$, or $\mathcal{R}_{RC}^{\mathcal{K}}(m) < \mathcal{R}_{RC}^{\mathcal{K}}(n)$, or $\mathcal{R}_{RC}^{\mathcal{K}}(m) = \mathcal{R}_{RC}^{\mathcal{K}}(n)$ and m satisfies more formulas than n in $\mathcal{K}$.*

This definition characterises lexicographic closure as a count-based refinement of rational closure. Its ranked model respects the rankings of rational closure (which encodes seriousness) but refines preference for worlds with the same rank based on the total number of formulas each satisfies.

3 Algorithm Development

Proofs for the propositions necessary to prove correctness of our algorithms can be found in the appendices.

3.1 ModelRank

Motivation The preference ordering over ranked interpretations in definition 4 characterises the minimal model with respect to other knowledge base models. We seek to develop an algorithm that directly constructs a representation of the minimal model without the need to compare models.

A way to view this problem is to consider starting with all the worlds as most preferred as possible and then performing only the most necessary 'bumping up' of worlds. Booth et al. [2, 3] take this approach in constructing what they refer to as the LM-minimum element for a Propositional Typicality Logic (PTL) knowledge base. Our initial algorithm makes use of a similar 'bumping up' approach. The intuition is to place as many worlds as possible on each rank to produce not only a model but the minimal ranked model with all the worlds as 'pushed down' as the knowledge permits [8].

We start with all the possible worlds for the propositional vocabulary of the knowledge base. Then, at each step of the algorithm, we place all the worlds that are models of the remaining materialised formulas from our knowledge base on the current rank. All such worlds are then removed from the collection of to-be-placed worlds to ensure they cannot be placed on more than one rank. Finally, we remove all the formulas whose antecedents are satisfied by a world we have just placed on the current rank from our collection of to-be-considered formulas. Together these two steps satisfy the requirements for minimal ranked entailment to hold.

Algorithm 4 `ModelRank`

Input: A defeasible knowledge base $\mathcal{K}$
Output: A ranked interpretation $(R_0, ..., R_{n-1}, R_\infty)$ and the number of ranks, n
$i := 0$;
$\mathcal{P}_\mathcal{K} := \{p \mid p$ is a propositional letter occurring in $\mathcal{K}\}$;
$\mathscr{U}_i :=$ universe of interpretations for vocabulary $\mathcal{P}_\mathcal{K}$;
$\mathcal{K}_i := \overrightarrow{\mathcal{K}}$;
repeat
    $R_i := \{v \in \mathscr{U}_i \mid v \Vdash \mathcal{K}_i\}$;
    $\mathscr{U}_{i+1} := \mathscr{U}_i \setminus R_i$;
    $\mathcal{K}_{i+1} := \{\alpha \to \beta \in \mathcal{K}_i \mid \nexists v \in R_i \text{ s.t. } v \Vdash \alpha\}$
    $i := i + 1$;
until $R_{i-1} = \emptyset$
$n := i - 1$
$R_\infty = \mathscr{U}_i$
return $(R_0, ..., R_{n-1}, R_\infty), n$

3.2 ModelRank Refinement

Motivation We wish to construct the ranked model corresponding to the lexicographic ordering in definition 6 using an approach similar to that of `ModelRank`.

Given the rational closure model and counts for each model, representing the number of formulas satisfied, there does not seem to be a straightforward way of directly computing the lexicographic rank of a valuation. Because the number of refined ranks produced from a rational closure rank varies, a less complicated strategy would be to employ a procedure that ranks valuations as necessary, removing the need to place the worlds directly. As the lexicographic ordering gives preference to valuations on lower rational closure ranks and satisfying more formulas, respectively, a simple approach would be to consider, for each of the rational closure ranks, in turn, every possible count of formulas that could be violated. For any combination of these two criteria the algorithm places all valuations, if any exist, that satisfy the criteria. This ensures that the relative order of worlds in the rational closure rank is maintained in the lexicographic model while refining based on formulas violated to produce the required ordering.

We formalize this bottom-up construction of the lexicographic ranked model in the `LexicographicModelRank` algorithm.

Algorithm 5 `LexicographicModelRank`

Input: A defeasible knowledge base $\mathcal{K}$
Output: An ordered tuple $(R_0^{LC}, ..., R_{k-1}^{LC}, R_\infty^{LC}, k)$
$(R_0^{RC}, ..., R_{n-1}^{RC}, R_\infty^{RC}, n) :=$ `ModelRank`$(\mathcal{K})$;
$i := 0$; ▷ rational closure rank
$k := 0$; ▷ lexicographic closure rank
while $i < n$ **do**
  $j := 0$; ▷ number of formulas to violate
  $\mathscr{U}_{ij} := R_i^{RC}$; ▷ remaining worlds to place
  while $\mathscr{U}_{ij} \neq \emptyset$ **do**
    $L_{ij} := \{u \in \mathscr{U}_{ij} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \not\Vdash k\} = j\}$;
    if $L_{ij} \neq \emptyset$ **then**
      $R_k^{LC} := L_{ij}$; ▷ place worlds violating j formulas
      $k := k + 1$;
    end if
    $\mathscr{U}_{i(j+1)} := \mathscr{U}_{ij} \setminus L_{ij}$; ▷ remove placed worlds
    $j := j + 1$;
  end while
  $i := i + 1$;
end while
$R_\infty^{LC} := R_\infty^{RC}$;
return $(R_0^{LC}, ..., R_{k-1}^{LC}, R_\infty^{LC}, k)$

Motivation The `ModelRank` algorithm directly produces the minimal ranked model in a representation consistent with its abstract definition in the literature [4, 8, 2]. Although that representation is suitable in an abstract setting, it is infeasible from an implementation standpoint. Furthermore, we note the space-complexity issues arising from the exponential relationship between the cardinality of the propositional vocabulary of a given knowledge base and the cardinality of the corresponding universe of worlds.

Therefore, we investigate new ways of representing ranked interpretations that still use the model-theoretic properties of minimal-ranked entailment but provide tractable alternatives to the current formula-based approaches.

Our first approach is then to construct formulas in correspondence with the levels of the rational closure model such that the models of each formula corresponds exactly with the worlds situated on the corresponding level in the rational closure model. That is, for a knowledge base, $\mathcal{K}$, and its corresponding minimal ranked model, $\mathcal{R}^{\mathcal{K}}_{RC} = (R_0, \cdots, R_{n-1}, R_\infty)$, we seek to construct a representation of the form $(F_0, \cdots, F_{n-1}, F_\infty)$, where each F_i is a propositional formula satisfying the condition: $Mod(F_i) = R_i$.

Hence, instead of enumerating the entire universe of worlds for the propositional vocabulary of the knowledge base and then determining whether such worlds satisfy specific criteria to place them on ranks, we instead 'place the criteria' itself on the ranks of the new representation.

3.3 FormulaRank

Algorithm 6 `FormulaRank`

1: Input: A defeasible knowledge base $\mathcal{K}$
2: Output: A ranked formula interpretation $(F_0, ..., F_{n-1}, F_\infty)$ and the number of ranks, n
3: $i := 0$;
4: $\mathcal{K}_i := \overrightarrow{\mathcal{K}}$;
5: **repeat**
6: $\quad F_i := (\bigwedge_i \mathcal{K}_i) \wedge \neg(\bigvee_{j<i} F_j)$;
7: $\quad \mathcal{K}_{i+1} := \{\alpha \to \beta \in \mathcal{K}_i \mid F_i \models \neg\alpha\}$;
8: $\quad i := i + 1$;
9: **until** $\mathcal{K}_i = \mathcal{K}_{i-1}$
10: $n := i$
11: $F_\infty := F_i$
12: **return** $(F_0, ..., F_{n-1}, F_\infty), n$

3.4 FormulaRank Refinement

Motivation The motivation for considering a formula-based lexicographic algorithm is precisely that for developing a formula-based rational closure algorithm. We take issue with the implementation of approaches that directly manipulate models, and for the same reasons outlined in 3.2, adapt our `LexicographicModelRank` algorithm to represent the models on each rank syntactically.

Using a formula-based version of the refinement strategy in `LexicographicModelRank`, we represent worlds satisfying a particular number of formulas n using all possible subsets of the knowledge with cardinality n. Combining these subsets disjunctively, we construct a formula with models satisfying at least n formulas, or equivalently, violating no more than $\#\mathcal{K} - n$ formulas. Therefore, when refining each rank, we start by constructing a formula with worlds on the rank violating no more than 0 formulas and, if any exist, removing these from the formula representing the remaining worlds. This process continues, as in `LexicographicModelRank`, until there are no remaining worlds, and is repeated for each rank.

Algorithm 7 `LexicographicFormulaRank`

Input: A defeasible knowledge base $\mathcal{K}$
Output: An ordered tuple $(F_0^{LC}, ..., F_{k-1}^{LC}, F_\infty^{LC}, k)$
$(F_0^{RC}, ..., F_{n-1}^{RC}, F_\infty^{RC}, n) :=$ `FormulaRank`$(\mathcal{K})$;
$i := 0$; ▷ rational closure rank
$k := 0$; ▷ lexicographic closure rank
while $i < n$ **do**
    $j := 0$; ▷ number of formulas to violate
    $\mathscr{U}_{ij} := F_i^{RC}$; ▷ remaining worlds to place
    while $\mathscr{U}_{ij} \not\models \bot$ **do**
        $L_{ij} := \mathscr{U}_{ij} \wedge \left(\bigvee_{S \in \{T \subseteq \overrightarrow{\mathcal{K}} \mid \#T = \#\overrightarrow{\mathcal{K}} - j\}} \bigwedge_{s \in S} s \right)$;
        if $L_{ij} \not\models \bot$ **then**
            $F_k^{LC} := L_{ij}$; ▷ place worlds violating j formulas
            $k := k + 1$;
        end if
        $\mathscr{U}_{i(j+1)} := \mathscr{U}_{ij} \wedge \neg L_{ij}$; ▷ remove placed worlds
        $j := j + 1$;
    end while
    $i := i + 1$;
end while
$F_\infty^{LC} := F_\infty^{RC}$;
return $(F_0^{LC}, ..., F_{k-1}^{LC}, F_\infty^{LC}, k)$

3.5 Cumulative FormulaRank

Motivation After implementing the `FormulaRank` algorithm using our extension of the Tweety Project Library, we encountered severe performance issues.

We determined the cause to be the interaction between the implementation of the Sat4j SAT solver [10] provided by the *TweetyProject* library [14] and the construction of the representative formulas on each rank.

Each representative rank formula comprises the conjunction of all the remaining formulas and the negation of the disjunction of all the previous representative rank formulas. The negation of the disjunction of all the previous representative rank formulas essentially asserts that we wish to exclude worlds that are already associated with the previous representative rank formulas.

We reformulated the entailment query of step seven of the `FormulaRank` algorithm to a satisfiability query that we can present to the SAT solver. The SAT solver then converts the given query to Conjunctive Normal Form (CNF) as part of its implementation. Hence, there is an exponential blowup in the number of clauses in the CNF of the original formula.

One can modify the definition of the representative rank formulas to no longer require the conjunction with the negation of the disjunction of all the previous representative rank formulas. The resulting sequence of formulas now represents an accumulation of worlds whereby the models of each formula are a subset of the models of the following formula. We term this new representation the 'cumulative ranked formula model' of a knowledge base. Significantly, this new representation does not affect our ability to answer entailment queries using minimal ranked entailment and avoids the complexity issues relating to the conversion to CNF.

This new representation is intimately related to the original `BaseRank` and `RationalClosure` algorithms. The `BaseRank` ranks are constructed from the difference between successive sets of exceptional formulas. `RationalClosure` answers entailment queries by starting with the union of all such `BaseRank` ranks and iteratively removing ranks from the lowest rank upwards until the antecedent of the query is classically consistent with the remaining knowledge. `RationalClosure` effectively reconstructs the sequence of exceptional sets initially produced by `BaseRank` from the `BaseRank` ranks to answer the entailment query.

We can show that the representative formula on a given finite rank of the cumulative ranked model is, in fact, the conjunction of the formulas in the exceptional set of the same index. Thus, not only does the cumulative ranked formula model provide a syntactic representation of the models of a given knowledge base in a cumulative sense, it functions as a cache of the information used by `RationalClosure` to answer entailment queries. Hence answering entailment queries using the cumulative ranked model is similar to the `RationalClosure` algorithm.

Algorithm 8 `CumulativeFormulaRank`

Input: A defeasible knowledge base $\mathcal{K}$
Output: A ranked formula interpretation $(F_0, ..., F_{n-1}, F_\infty)$ and the number of ranks, n
$i := 0;$
$\mathcal{K}_i := \overrightarrow{\mathcal{K}};$
repeat
    $F_i := (\bigwedge \mathcal{K}_i);$
    $\mathcal{K}_{i+1} := \{\alpha \to \beta \in \mathcal{K}_i \mid F_i \models \neg\alpha\};$
    $i := i + 1;$
until $\mathcal{K}_i = \mathcal{K}_{i-1}$
$n := i$
$F_\infty := F_i$
return $(F_0, ..., F_{n-1}, F_\infty), n$

4 Lexicographic Closure

We wish to formulate a cumulative approach for computing lexicographic closure that highlights the relationship between the model-theoretic definition and the usual `LexicographicClosure` algorithm. Based on our cumulative rational closure approach findings, we expect that the cumulative model ranks correspond to various iterations of the original `LexicographicClosure` algorithm. In attempting such, however, we make an important observation regarding the distinction between definitions of lexicographic closure presented in [11] and [4].

The refinement definition of lexicographic closure applied in our model-based algorithms defines an ordering based on the criteria of seriousness and count (similar to the ordering defined by Lehmann). However, we find that this is, in fact, distinct from the ordering originally defined for lexicographic closure in [11].

We prove, via an example, how these definitions differ regarding the produced ranked models.

Example 1. Consider $\mathcal{K} = \{p \to b, b \mid\!\sim f, b \mid\!\sim w, p \mid\!\sim \neg f, p \mid\!\sim w\}$.

This represents the knowledge that all penguins are birds, birds typically fly, birds typically have wings, penguins typically don't fly, and penguins typically have wings.

The base rank of the formulas in $\mathcal{K}$ and the corresponding minimal ranked (rational closure) model are shown in figure 1.

We construct the ranked models in figure 2 according to the, purportedly equivalent, Lehmann [11] and Casini et al. [4] definitions of lexicographic closure.

We observe that both lexicographic models respect the relative order of the worlds in the rational closure model.

However, we notice a difference in the refinement of the second rational closure rank in producing the two lexicographic models. In particular, consider valuations $\mathtt{bfp\overline{w}}$ and $\mathtt{b\overline{f}p\overline{w}}$(circled in the rational closure model). The tuples of

∞	$p \rightarrow b$
1	$p \mathrel{\mid\!\sim} \neg f, p \mathrel{\mid\!\sim} w$
0	$b \mathrel{\mid\!\sim} f, b \mathrel{\mid\!\sim} w$

(a) Base Rank

∞	$\mathtt{\overline{b}fp\overline{w}}$ $\mathtt{\overline{bf}pw}$ $\mathtt{\overline{bfp}\,\overline{w}}$ $\mathtt{\overline{b}fpw}$
2	$\mathtt{bfpw}$ $\mathtt{bfp\overline{w}}$ $\mathtt{b\overline{f}p\overline{w}}$
1	$\mathtt{bf\overline{pw}}$ $\mathtt{b\overline{f}\,\overline{p}w}$ $\mathtt{b\overline{f}\,\overline{pw}}$ $\mathtt{b\overline{f}pw}$
0	$\mathtt{bf\overline{p}w}$ $\mathtt{\overline{b}f\overline{p}w}$ $\mathtt{\overline{b}f\overline{pw}}$ $\mathtt{\overline{bf}\,\overline{p}w}$ $\mathtt{\overline{bf}\,\overline{pw}}$

(b) Minimal Ranked Model

Fig. 1: Base Rank and Minimal Ranked Model of $\mathcal{K}$

∞	$\mathtt{\overline{b}fp\overline{w}}$ $\mathtt{\overline{bf}pw}$ $\mathtt{\overline{bfp}\,\overline{w}}$ $\mathtt{\overline{b}fpw}$
5	$\mathtt{bfp\overline{w}}$
4	$\mathtt{b\overline{f}p\overline{w}}$
3	$\mathtt{bfpw}$
2	$\mathtt{b\overline{f}\,\overline{pw}}$
1	$\mathtt{bf\overline{pw}}$ $\mathtt{b\overline{f}\,\overline{p}w}$ $\mathtt{b\overline{f}pw}$
0	$\mathtt{bf\overline{p}w}$ $\mathtt{\overline{b}f\overline{p}w}$ $\mathtt{\overline{b}f\overline{pw}}$ $\mathtt{\overline{bf}\,\overline{p}w}$ $\mathtt{\overline{bf}\,\overline{pw}}$

(a) Lexicographic Closure [11] Model

∞	$\mathtt{\overline{b}fp\overline{w}}$ $\mathtt{\overline{bf}pw}$ $\mathtt{\overline{bfp}\,\overline{w}}$ $\mathtt{\overline{b}fpw}$
4	$\mathtt{bfp\overline{w}}$ $\mathtt{b\overline{f}p\overline{w}}$
3	$\mathtt{bfpw}$
2	$\mathtt{b\overline{f}\,\overline{pw}}$
1	$\mathtt{bf\overline{pw}}$ $\mathtt{b\overline{f}\,\overline{p}w}$ $\mathtt{b\overline{f}pw}$
0	$\mathtt{bf\overline{p}w}$ $\mathtt{\overline{b}f\overline{p}w}$ $\mathtt{\overline{b}f\overline{pw}}$ $\mathtt{\overline{bf}\,\overline{p}w}$ $\mathtt{\overline{bf}\,\overline{pw}}$

(b) Lexicographic Closure [4] Model

Fig. 2: The Two Lexicographic Ranked Models

violated formula counts ordered by base rank, as defined in the Lehmann definition of lexicographic closure [11], are $\langle 0, 2, 1\rangle$ and $\langle 0, 1, 2\rangle$, respectively. We define similar tuples based on the refinement criteria of the Casini et al. lexicographic ordering [4] to include the rational closure rank and the number of formulas violated in $\mathcal{K}$ (as the ordering favours valuations with a lower rational closure rank, violating fewer formulas in $\mathcal{K}$). Both valuations, in this case, have the same rational closure rank of two and violate three formulas in $\mathcal{K}$. Hence the corresponding tuple associated with these valuations is $\langle 2, 3\rangle$. Noting that both partial orders can be defined by comparing the corresponding tuples using the natural lexicographic ordering of tuples, the Lehmann ordering places the valuations on different ranks while the Casini et al. ordering places them on the same rank.

Example 1 demonstrates that refining the rational closure model ranks by count alone does not necessarily produce the Lehmann lexicographic closure ranked model. The rational closure model separates valuations according to the number of trailing zeroes in their formula violation tuples or, equivalently, the highest base rank among the formulas each violates (based on the connection between the base rank of a formula and the rank of its minimal world in the minimal ranked model [7]). Therefore, to refine such to produce the Lehmann lexicographic closure model, valuations on the same rational closure rank must be separated, not based on the number of formulas violated in total [4], but rather by considering each of the remaining violation counts in turn (essentially completing the lexicographic tuple comparison).

Nonetheless, the ordering defined in [4] still represents a valid form of lexicographic closure in which the tuples used to compare valuations are of order two, consisting of a valuation's rational closure rank and formula violation count. Notably, the ranked models corresponding to each ordering constitute refinements of the rational closure model and will, therefore, fall within the rational defeasible entailment framework described in [4].

While it would be possible to construct a cumulative version of the `LexicographicFormulaRank` algorithm by instead refining cumulative rational closure ranks without the negation of prior ranks (much like in the `CumulativeFormulaRank` algorithm), we would need to show that such represents the cumulative model. The output of the `CumulativeFormulaRank` algorithm is cumulative as the formulas on each rank become strictly weaker as the rank increases. While such is also the case for iterations of the `LexicographicClosure` algorithm, it may not hold for the modified `LexicographicFormulaRank`, as described above. Therefore, a cumulative algorithm may need to explicitly include the prior cumulative rational closure ranks, as it is possible for a lower-ranked valuation to violate more formulas than a higher-ranked valuation.

5 Complexity Analysis

We assume a propositional vocabulary containing p atoms and hence 2^p possible worlds. We claim that 2^p satisfaction checks can be considered approximately equivalent to a single entailment check as, in the worst case, checking whether a particular entailment, $\alpha \models \beta$, holds can be evaluated by determining whether $\alpha \wedge \neg\beta$ is unsatisfiable.

We estimate the space complexity of `ModelRank` and `LexicographicModelRank` in terms of the number of propositional worlds to be stored. For the remaining algorithms, we estimate their space complexity in terms of the number of syntactic knowledge base formulas their resulting representations comprise. This decision is motivated by the fact that the representative formulas on each rank entirely comprise combinations of formulas from the original knowledge base.

Time complexity results in table 1 show that all algorithms perform in the order of n^2 classical entailment checks and are thus not much more complex than the problem of boolean satisfiability for propositional logic (solvable in non-deterministic polynomial time) [12].

Under the assumption that storage requirements for valuations and formulas do not differ significantly, we favour the space efficiency of the `CumulativeFormulaRank` algorithm for constructing a representation of the rational closure model. However, the difficulty of expressing counts in propositional logic produces a super-exponential space complexity for the `LexicographicFormu-`

`laRank`. We, therefore, favour the `LexicographicModelRank` algorithm despite its exponential space complexity.

Algorithm	Time	Space
`ModelRank`	$O(n^2)$	$O(2^p)$
`FormulaRank`	$O(n^2)$	$O(2^n \times n^2)$
`CumulativeFormulaRank`	$O(n^2)$	$O(n^2)$
`LexicographicModelRank`	$O(n^2)$	$O(2^p)$
`LexicographicFormulaRank`	$O(n^2)$	$O(2^n \times n^3)$

Table 1: Algorithm Time and Space Complexities

6 Conclusions and Future Work

Our work represents an avenue largely unexplored in the literature: the design of model-based algorithms for computing forms of KLM-style defeasible entailment.

We present five new algorithms for constructing representations of the rational and lexicographic closure ranked models of a given defeasible knowledge base. The first two construct representations consistent with those abstractly defined elsewhere in the literature. The remaining three construct new compact representations for the ranked models using representative formulas. The thrid rational closure algorithm produces a new class of representation that we term cumulative, as the models of each rank's representative formula are precisely those on and below the corresponding rational closure rank.

With all algorithms following the same bottom-up pattern of construction, based on the initial model ranking algorithms, we prove these produce the desired ranked models for rational and lexicographic closure.

In attempting to formulate a cumulative algorithm for lexicographic closure, we find that the ordering defined in [4] for lexicographic closure differs from that initially described in [11]. While both constitute refinements of the rational closure model, they represent distinct forms of reasoning that will need to be compared and further explored.

In light of this observation, we need to develop similar algorithms for the Lehmann lexicographic closure and a more compact representation for the Casini et al. lexicographic closure.

Additionally, we wish to explore whether these algorithms and their corresponding model representations may be generalised to compute any rational defeasible entailment relation [4].

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A `ModelRank` Algorithm Proofs

Proposition 1. *The `ModelRank` algorithm terminates.*

Proof. We assume that $\overrightarrow{\mathcal{K}}$ is consistent. Thus, we want to show that $R_i = \emptyset$ for some i.

By definition, we have that $\forall j, R_j := \mathscr{U}_j \cap Mod(\mathcal{K}_j)$ and $\mathscr{U}_{j+1} \subseteq \mathscr{U}_j$.

Thus for arbitrary i, either:

1. $\mathscr{U}_{i+1} \subset \mathscr{U}_i$
2. $\mathscr{U}_{i+1} = \mathscr{U}_i$

Since $\mathscr{U}$ is finite, (1) can only occur a finite number of times. If $\mathscr{U}_{i+1} = \mathscr{U}_i$, then $R_i = \emptyset$ since $\mathscr{U}_{i+1} := \mathscr{U}_i \setminus R_i$ and $R_i \subseteq \mathscr{U}_i$.

Proposition 2. *The `ModelRank` algorithm produces a ranked model of the given defeasible knowledge base, $\mathcal{K}$.*

Proof. Suppose `ModelRank` produces $\mathcal{R}^* = (R_0, \cdots, R_{n-1}, R_\infty)$.

We therefore wish to show that $\mathcal{R}^*$ is a ranked model of $\mathcal{K}$.

We show this in two parts:

1. **$\mathcal{R}^*$ a ranked interpretation:**
   We show that $\mathcal{R}^*$ is a function from $\mathscr{U}$ to $\mathbb{N} \cup \{\infty\}$ such that $\mathcal{R}^*(u) = 0$ for some $u \in \mathscr{U}$, and satisfying the following convexity property: $\forall i \in \mathbb{N}$, if $\mathcal{R}^*(v) = i$, then, for $\forall j$ such that $0 \leq j < i$, $\exists u \in \mathscr{U}$ for which $\mathcal{R}^*(u) = j$.
   We assume that $\overrightarrow{\mathcal{K}}$ is consistent.
   Hence, $Mod(\overrightarrow{\mathcal{K}}) \neq \emptyset$. Thus $R_0 := \mathscr{U}_0 \cap Mod(\mathcal{K}_0) \neq \emptyset$.
   Thus, $\exists u \in \mathscr{U}$ such that $\mathcal{R}^*(u) = 0$.
   Take arbitrary $u \in \mathscr{U}$.
   Either $u \in R_j$ or $u \notin R_j$ for some $j \in \mathbb{N}$.
   If $u \in R_j$, then $u \in \mathscr{U}_j$ and $u \in Mod(\mathcal{K}_j)$.
   $$\begin{aligned} & u \in \mathscr{U}_j \text{ and } u \in R_j \\ & \Rightarrow u \notin \mathscr{U}_{j+1} \\ & \Rightarrow \forall m > 0, u \notin \mathscr{U}_{j+1+m}, \text{ since } \forall i > 0, \mathscr{U}_{i+1} \subset \mathscr{U}_i \\ & \Rightarrow \forall m > 0, u \notin R_{j+1+m} \end{aligned}$$
   If $u \notin R_j$ for some $j \in \mathbb{N}$, then $u \in \mathscr{U}_i$ for $i \leq n$. But $R_\infty := \mathscr{U}_n$, thus $u \in R_\infty$.
   Thus, as soon as a world is placed on a rank, it can no longer be placed on any subsequent ranks. We note that the stopping condition of the algorithm is that the current rank is empty, and that this empty rank is excluded from the output. Thus, there can never be any empty ranks.
2. **$\mathcal{R}^*$ is a model of $\mathcal{K}$:**
   We want to show that $\forall \alpha \mid\!\sim \beta \in \mathcal{K}$, $min_\prec [\![\alpha]\!]^{\mathcal{R}^*} \subseteq [\![\beta]\!]^{\mathcal{R}^*}$.
   We note that $min_\prec [\![\alpha]\!]^{\mathcal{R}^*}$ is just alternative notation for the minimal α-worlds with respect to the interpretation $\mathcal{R}^*$.
   Take arbitrary $\alpha \mid\!\sim \beta \in \mathcal{K}$.

(a) If $[\![\alpha]\!]^{\mathcal{R}^*} = \emptyset$, we are done.
(b) If $[\![\alpha]\!]^{\mathcal{R}^*} \neq \emptyset$, then take arbitrary $v \in [\![\alpha]\!]^{\mathcal{R}^*}$.
Suppose $\mathcal{R}^*(v) = i$.
Note that $R_j := \mathscr{U}_j \cap Mod(\mathcal{K}_j)$ and

$$\mathcal{K}_j := \{\gamma \to \delta \in \mathcal{K}_{j-1} \mid \nexists v \in R_{j-1} \text{ s.t. } v \Vdash \gamma\}.$$

Since $v \in [\![\alpha]\!]^{\mathcal{R}^*}$ and $\mathcal{R}^*(v) = i$, we must have that $\forall j < i$, $\nexists u \in R_j$ such that $u \Vdash \alpha$.
Note that $\alpha \to \beta \in \mathcal{K}_0 := \overrightarrow{\mathcal{K}}$.
Hence, $\alpha \to \beta \in \mathcal{K}_j$, $\forall j \leq i$.
Since $v \in [\![\alpha]\!]^{\mathcal{R}^*}$ and $v \in R_i := \mathscr{U}_i \cap Mod(\mathcal{K}_i)$ and $\alpha \to \beta \in \mathcal{K}_i$, we have that $v \in [\![\beta]\!]^{\mathcal{R}^*}$.

Lemma 1. *Suppose* `ModelRank` *produces* $\mathcal{R}^* = (R_0, \cdots, R_{n-1}, R_\infty)$.
Take arbitrary $v \in R_i$ *for* $i > 0$.
$\forall \alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$, $\exists w \in R_{i-1}$, *s.t.* $w \Vdash \alpha \wedge \beta$.

Proof. Take arbitrary $\alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$

$$\begin{aligned}
\Rightarrow \alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i &= \mathcal{K}_{i-1} \cap \overline{\mathcal{K}_i} \\
&= \mathcal{K}_{i-1} \cap \overline{\{\alpha \to \beta \in \mathcal{K}_{i-1} \mid \nexists v \in R_i \text{ s.t. } v \Vdash \alpha\}} \\
&= \mathcal{K}_{i-1} \cap \overline{(\mathcal{K}_{i-1} \cap \{\alpha \to \beta \in \overrightarrow{\mathcal{K}} \mid \nexists v \in R_i \text{ s.t. } v \Vdash \alpha\})} \\
&= \mathcal{K}_{i-1} \cap (\overline{\mathcal{K}_{i-1}} \cup \overline{\{\alpha \to \beta \in \overrightarrow{\mathcal{K}} \mid \nexists v \in R_i \text{ s.t. } v \Vdash \alpha\}}) \\
&= \mathcal{K}_{i-1} \cap \overline{\{\alpha \to \beta \in \overrightarrow{\mathcal{K}} \mid \nexists v \in R_i \text{ s.t. } v \Vdash \alpha\}} \\
&= \mathcal{K}_{i-1} \cap \{\alpha \to \beta \in \overrightarrow{\mathcal{K}} \mid \exists v \in R_i \text{ s.t. } v \Vdash \alpha\} \\
&= \{\alpha \to \beta \in \mathcal{K}_{i-1} \mid \exists v \in R_i \text{ s.t. } v \Vdash \alpha\}
\end{aligned}$$

Since $\alpha \to \beta \in \{\alpha \to \beta \in \mathcal{K}_{i-1} \mid \exists v \in R_i \text{ s.t. } v \Vdash \alpha\}$, take arbitrary $u \in R_{i-1}$ such that $u \Vdash \alpha$.
But $u \in R_i \Rightarrow u \in Mod(\mathcal{K}_{i-1}) \subseteq Mod(\mathcal{K}_{i-1} \setminus \mathcal{K}_i)$.
Hence, $u \Vdash \alpha$ and $u \in Mod(\mathcal{K}_{i-1} \setminus \mathcal{K}_i)$ and $\alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i \Rightarrow u \Vdash \beta$.

Lemma 2. *Suppose* `ModelRank` *produces* $\mathcal{R}^* = (R_0, \cdots, R_{n-1}, R_\infty)$.
Let $i > 0$.
$\forall \alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i, \forall j < i - 1, \nexists w \in R_j$, *s.t.* $w \Vdash \alpha$.

Proof. Take arbitrary $\alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$.
Suppose for the sake of contradiction that $\exists w \in R_j$ with $j < i - 1$ and that $w \Vdash \alpha$.
By definition, $R_j = \mathscr{U}_j \cap Mod(\mathcal{K}_j)$.
$\Rightarrow w \in Mod(\mathcal{K}_j)$

By definition, $\forall m > 0$, $\mathcal{K}_{m+1} \subset \mathcal{K}_m$.
By assumption, $\alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i \subseteq \mathcal{K}_{i-1}$
$\Rightarrow \alpha \to \beta \in \mathcal{K}_{i-1} \subset \cdots \subset \mathcal{K}_j \subset \mathcal{K}_{j-1} \subset \cdots \subset \mathcal{K}_1 \subset \mathcal{K}_0 := \overrightarrow{\mathcal{K}}$.
Note that $\mathcal{K}_{j+1} := \{\alpha \to \beta \in \mathcal{K}_j \mid \nexists v \in R_j \text{ s.t. } v \Vdash \alpha\}$.
Now, $\alpha \to \beta \in \mathcal{K}_j$ and $w \in R_j$ and $w \Vdash \alpha$.
Hence, $\alpha \to \beta \notin \mathcal{K}_{j+1} \Rightarrow \alpha \to \beta \notin \mathcal{K}_{i-1} \Rightarrow \alpha \to \beta \notin \mathcal{K}_{i-1} \setminus \mathcal{K}_i$.
Which is clearly a contradiction. Thus no such w exists.

Lemma 3. *Suppose* `ModelRank` *produces* $\mathcal{R}^* = (R_0, \cdots, R_{n-1}, R_\infty)$.
Take arbitrary $v \in R_i$ *for* $i > 0$.
$\forall \alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$, $min_\prec [\![\alpha \wedge \beta]\!]^{\mathcal{R}^*} \subseteq R_{i-1}$.

Proof. This follows from Lemma 1 and Lemma 2 since Lemma 1 shows that there exists a world with the specified property and Lemma 2 shows that there does not exist a world with such property on any lower rank.

Lemma 4. *Suppose* `ModelRank` *produces* $\mathcal{R}^* = (R_0, \cdots, R_{n-1}, R_\infty)$.
Take arbitrary $v \in R_i$ *for* $i > 0$.
$\exists \alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$ *such that* $v \nVdash \alpha \to \beta$

Proof. Suppose for the sake of contradiction that $v \in R_i$ and $\forall \alpha \to \beta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$, $v \Vdash \alpha \to \beta$.
Hence, $v \in Mod(\mathcal{K}_{i-1} \setminus \mathcal{K}_i)$.
By definition, $R_i = \mathscr{U}_i \cap Mod(\mathcal{K}_i) \subseteq Mod(\mathcal{K}_i)$.
$\Rightarrow v \in \mathscr{U}_i$ and $v \in Mod(\mathcal{K}_i)$.
Take arbitrary $x \in \mathcal{K}_{i-1}$.
Since $\mathcal{K}_i \subset \mathcal{K}_{i-1}$, we have that $\mathcal{K}_{i-1} = \mathcal{K}_i \cup (\mathcal{K}_{i-1} \setminus \mathcal{K}_i)$.
Hence, either $x \in \mathcal{K}_i$ or $x \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$.
If $x \in \mathcal{K}_i$, then since $v \in Mod(\mathcal{K}_i)$, $v \Vdash x$.
If $x \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i$, then since $v \in Mod(\mathcal{K}_{i-1} \setminus \mathcal{K}_i)$, $v \Vdash x$.
Thus $v \in Mod(\mathcal{K}_{i-1})$.
Hence, $v \in \mathscr{U}_{i-1}$ and $v \in Mod(\mathcal{K}_{i-1}) \Rightarrow v \in R_{i-1}$.
This is a contradiction since we assumed that $v \in R_i$ and we have shown that $\mathcal{R}^*$ is a ranked interpretation.

Consider the ordering $\preceq_\mathcal{K}$ on all ranked models of a knowledge base $\mathcal{K}$, which is defined as follows: $\mathcal{R}_1 \preceq_\mathcal{K} \mathcal{R}_2$ if for every $v \in \mathscr{U}$, $\mathcal{R}_1(v) \leq \mathcal{R}_2(v)$.

Proposition 3. *Suppose* `ModelRank` *produces* $\mathcal{R}^* = (R_0, \cdots, R_{n-1}, R_\infty)$. $\mathcal{R}^*$ *is the minimal ranked model of* $\mathcal{K}$ *with respect to* $\preceq_\mathcal{K}$.

Proof. Suppose `ModelRank` produces $\mathcal{R}^* = (R_0, \cdots, R_{n-1}, R_\infty)$.
Take arbitrary $v \in R_i$ for $i > 0$.
We want to show that if we remove v and place it on any rank lower than i, that the resulting ranked interpretation, is no longer a model of $\mathcal{K}$.
To do this, we use Lemma 3 and Lemma 4.

Lemma 3 shows that all the best alpha worlds, that are also beta worlds, for any formula in $\mathcal{K}_{i-1} \setminus \mathcal{K}_i$, are located on rank $i-1$.

Lemma 4 then shows that there must be at least one formula, say $\gamma \to \delta$, in $\mathcal{K}_{i-1} \setminus \mathcal{K}_i$ that v violates.

Hence, $\gamma \to \delta \in \mathcal{K}_{i-1} \setminus \mathcal{K}_i \subset \mathcal{K}_{i-1} \subset \cdots \subset \mathcal{K}_0$.

Thus, we can conclude that

$$\mathcal{R}^{*'} := (R_0, \cdots, R_{i-k} \cup \{v\}, \cdots, R_i \setminus \{v\}, \cdots, R_{n-1}, R_\infty)$$

for some $0 < k \leq i$ is not a model of $\mathcal{K}$.

B `FormulaRank` Algorithm Proofs

Lemma 5. *Consider each set of remaining worlds, $\mathscr{U}_i$, as defined in the `ModelRank` algorithm as $\mathscr{U}_i := \mathscr{U}_{i-1} \setminus R_{i-1}, \forall i > 0$. One can write, $\forall i > 0, \mathscr{U}_i = \mathscr{U} \setminus \bigcup_{j=0}^{i-1} R_j$.*

Proof. We use induction.

- Base Case:

$$\begin{aligned} \mathscr{U}_1 &= \mathscr{U}_0 \setminus R_0 && \text{(by definition)} \\ &= \mathscr{U}_0 \setminus \bigcup_{j=0}^{0} R_j \end{aligned}$$

- Induction Step: suppose for some $k > 0, k \in \mathbb{N}$ that

$$\mathscr{U}_k = \mathscr{U} \setminus \bigcup_{j=0}^{k-1} R_j \text{ holds.}$$

We wish to show that

$$\mathscr{U}_{k+1} = \mathscr{U} \setminus \bigcup_{j=0}^{k} R_j$$

$$\begin{aligned}
\mathscr{U}_{k+1} &= \mathscr{U}_k \setminus R_k && \text{(by definition)}\\
&= (\mathscr{U} \setminus \bigcup_{j=0}^{k-1} R_j) \setminus R_k\\
&= \mathscr{U} \setminus ((\bigcup_{j=0}^{k-1} R_j) \cup R_k)\\
&= \mathscr{U} \setminus \bigcup_{j=0}^{k} R_j
\end{aligned}$$

Proposition 4. *With respect to the* `ModelRank` *and* `FormulaRank` *algorithms, the representative formula, F_i, on each rank of the* `FormulaRank` *model, is related to the worlds on each rank, R_i, of the* `ModelRank` *model by the following property: $\forall i,\ Mod(F_i) = R_i$ and $\mathcal{K}'_i = \mathcal{K}_i$. Additionally, both algorithms terminate at the same point.*

Proof. Base Case:

We assume that $\mathcal{K}$ is consistent.
Thus we have that $R_0 := \mathscr{U}_0 \cap Mod(\mathcal{K}_0) = Mod(\overrightarrow{\mathcal{K}})$ is not empty.
Furthermore, for both `ModelRank` and `FormulaRank`, $\mathcal{K}_0 := \overrightarrow{\mathcal{K}}$ and $\mathcal{K}'_0 := \overrightarrow{\mathcal{K}}$.
Thus $\mathcal{K}_0 = \mathcal{K}'_0$.

$$\begin{aligned}
F_0 &:= \bigwedge \mathcal{K}'_0 \wedge \neg(\bigvee_{j<0} F_j)\\
&= \bigwedge \mathcal{K}'_0 \wedge \neg\bot\\
&= \bigwedge \mathcal{K}'_0 \wedge \top\\
&= \bigwedge \mathcal{K}'_0\\
&= \bigwedge \mathcal{K}_0 && \text{(by definition)}\\
\Rightarrow Mod(F_0) &= Mod(\bigwedge \mathcal{K}_0)\\
&= Mod(\mathcal{K}_0)\\
&= \mathscr{U}_0 \cap Mod(\mathcal{K}_0) && \text{(since } \mathscr{U}_0 := \mathscr{U}\ \text{)}\\
&= R_0
\end{aligned}$$

We also know that both $\mathcal{K}_1$ and $\mathcal{K}'_1$ exist.

'Repeating Base Case':

Suppose that for some $i > 0$, $R_i \neq \emptyset$, that $\mathcal{K}_i = \mathcal{K}'_i$ and that $\forall k \leq i$, $Mod(F_k) = R_k$.

We first show that $\mathcal{K}_{i+1} = \mathcal{K}'_{i+1}$.

Note that

$$\begin{aligned}&\mathcal{K}_{i+1} := \{\alpha \to \beta \in \mathcal{K}_i \mid \nexists v \in R_i s.t. v \Vdash \alpha\}\\ &\text{ and}\\ &\mathcal{K}'_{i+1} := \{\alpha \to \beta \in \mathcal{K}'_i \mid F_i \models \neg\alpha\}.\end{aligned}$$

Since $\mathcal{K}_i = \mathcal{K}'_i$ by our induction hypothesis,

$$\mathcal{K}'_{i+1} = \{\alpha \to \beta \in \mathcal{K}_i \mid F_i \models \neg\alpha\}.$$

Now,

$$\begin{aligned}F_i \models \neg\alpha &\Leftrightarrow Mod(F_i) \subseteq Mod(\neg\alpha)\\ &\Leftrightarrow R_i \subseteq Mod(\neg\alpha) && \text{(by induction hypothesis)}\\ &\Leftrightarrow \forall u \in R_i, u \in Mod(\neg\alpha)\\ &\Leftrightarrow \forall u \in R_i, u \Vdash \neg\alpha\\ &\Leftrightarrow \nexists u \in R_i, u \Vdash \alpha\end{aligned}$$

Thus, since $\mathcal{K}_i = \mathcal{K}'_i$ (by induction hypothesis), and $F_i \models \neg\alpha \Leftrightarrow \nexists u \in R_i$ s.t. $u \Vdash \alpha$, we have that $\mathcal{K}_{i+1} = \mathcal{K}'_{i+1}$.

Now,

$$\begin{aligned}F_{i+1} &:= \bigwedge \mathcal{K}'_{i+1} \wedge \neg(\bigvee_{j<i+1} F_j)\\ &= \bigwedge \mathcal{K}_{i+1} \wedge \neg(\bigvee_{j<i+1} F_j)\\ \Rightarrow Mod(F_{i+1}) &= Mod(\bigwedge \mathcal{K}_{i+1}) \cap Mod(\neg(\bigvee_{j<i+1} F_j))\end{aligned}$$

Now,

$$\begin{aligned}Mod(\neg(\bigvee_{j<i+1} F_j)) &= \mathscr{U} \setminus Mod(\bigvee_{j<i+1} F_j)\\ &= \mathscr{U} \setminus \bigcup_{j=0}^{i} Mod(F_j)\\ &= \mathscr{U} \setminus \bigcup_{j=0}^{i} R_j && \text{(by induction hypothesis)}\\ &= \mathscr{U}_{i+1} && \text{(by lemma 5)}\end{aligned}$$

Thus,

$$\begin{aligned}
Mod(F_{i+1}) &= Mod(\bigwedge \mathcal{K}_{i+1}) \cap \mathscr{U}_{i+1} \\
&= \mathscr{U}_{i+1} \cap Mod(\mathcal{K}_{i+1}) \\
&= R_{i+1} && \text{(by definition)}
\end{aligned}$$

Now, if $\mathcal{K}'_{i+1} = \mathcal{K}'_i$, then we have that `FormulaRank` terminates.
We must now show that `ModelRank` terminates at the same index $(i+1)$.

$$\begin{aligned}
R_{i+1} :&= \mathscr{U}_{i+1} \cap Mod(\mathcal{K}_{i+1}) \\
&= \mathscr{U}_{i+1} \cap Mod(\mathcal{K}_i) \\
&= (\mathscr{U}_i \setminus R_i) \cap Mod(\mathcal{K}_i) \\
&= (\mathscr{U}_i \cap Mod(\mathcal{K}_i)) \setminus (R_i \cap Mod(\mathcal{K}_i)) \\
&= R_i \setminus R_i \\
&= \emptyset
\end{aligned}$$

Thus `ModelRank` terminates at the same index.

C `CumulativeFormulaRank` Algorithm Proofs

Proposition 5. *With respect to the* ***`ModelRank`*** *and* ***`CumulativeFormulaRank`*** *algorithms, the representative formula, F'_i, on each rank of the* ***`CumulativeFormulaRank`*** *model, is related to the worlds on each rank, R_i, of the* ***`ModelRank`*** *model by the following property: $\forall i$, $Mod(F'_i) = \bigcup_{j=0}^{i} R_j$ and $\mathcal{K}'_i = \mathcal{K}_i$. Additionally, both algorithms terminate at the same point.*

Proof. Base Case:

We assume that $\mathcal{K}$ is consistent.
Thus we have that $R_0 := \mathscr{U}_0 \cap Mod(\mathcal{K}_0) = Mod(\overrightarrow{\mathcal{K}})$ is not empty.
Furthermore, for both ModelRank and FormulaRank, $\mathcal{K}_0 := \overrightarrow{\mathcal{K}}$ and $\mathcal{K}'_0 := \overrightarrow{\mathcal{K}}$.
Thus $\mathcal{K}_0 = \mathcal{K}'_0$.

$$\begin{aligned}
F'_0 &:= \bigwedge \mathcal{K}'_0 \\
&= \bigwedge \mathcal{K}_0 && \text{(by definition)} \\
\Rightarrow Mod(F_0) &= Mod(\bigwedge \mathcal{K}_0) \\
&= Mod(\mathcal{K}_0) \\
&= \mathscr{U}_0 \cap Mod(\mathcal{K}_0) && \text{(since } \mathscr{U}_0 := \mathscr{U}\text{)} \\
&= R_0 \\
&= \bigcup_{j=0}^{0} R_j
\end{aligned}$$

We also know that both $\mathcal{K}_1$ and $\mathcal{K}'_1$ exist.

'Repeating Base Case':

Suppose for that for some $i > 0$, $R_i \neq \emptyset$, that $\mathcal{K}_i = \mathcal{K}'_i$ and that $\forall k \leq i$, $Mod(F'_k) = \bigcup_{j=0}^{k} R_j$.

We first show that $\mathcal{K}_{i+1} = \mathcal{K}'_{i+1}$.

Note that

$$\begin{aligned}
&\mathcal{K}_{i+1} := \{\alpha \to \beta \in \mathcal{K}_i \mid \nexists v \in R_i s.t. v \Vdash \alpha\} \\
&\text{ and} \\
&\mathcal{K}'_{i+1} := \{\alpha \to \beta \in \mathcal{K}'_i \mid F'_i \models \neg\alpha\}.
\end{aligned}$$

Since $\mathcal{K}_i = \mathcal{K}'_i$ by our induction hypothesis,

$$\mathcal{K}'_{i+1} = \{\alpha \to \beta \in \mathcal{K}_i \mid F_i \models \neg\alpha\}.$$

We show that $\mathcal{K}_{i+1} \subseteq \mathcal{K}'_{i+1}$ and $\mathcal{K}'_{i+1} \subseteq \mathcal{K}_{i+1}$.
If $\mathcal{K}'_{i+1} \neq \emptyset$, then take arbitrary $\alpha \to \beta \in \mathcal{K}'_{i+1}$. Thus, we have that

$$\begin{aligned}
F'_i \models \neg\alpha &\Leftrightarrow Mod(F'_i) \subseteq Mod(\neg\alpha) \\
&\Leftrightarrow \bigcup_{j=0}^{i} R_j \subseteq Mod(\neg\alpha) \\
&\Rightarrow R_i \subseteq Mod(\neg\alpha) \\
&\Rightarrow \alpha \to \beta \in \mathcal{K}_{i+1}
\end{aligned}$$

If $\mathcal{K}_{i+1} \neq \emptyset$, then take arbitrary $\alpha \to \beta \in \mathcal{K}_{i+1}$. Thus, we have that

$$\begin{aligned}\nexists v \in R_i, \text{ s.t. } v \Vdash \alpha &\Rightarrow \nexists v \in R_j, \forall j \leq i \text{ s.t. } v \Vdash \alpha \\ &\Rightarrow \bigcup_{j=0}^{i} R_j \subseteq Mod(\neg\alpha) \\ &\Rightarrow Mod(F'_i) \subseteq Mod(\neg\alpha) \\ &\Leftrightarrow F'_i \models \neg\alpha \\ &\Rightarrow \alpha \to \beta \in \mathcal{K}'_{i+1}\end{aligned}$$

Thus $\mathcal{K}_{i+1} = \mathcal{K}'_{i+1}$.

We now need to show that $Mod(F'_{i+1}) = \bigcup_{j=0}^{i+1} R_j$.

We first show that $\bigcup_{j=0}^{i+1} R_j \subseteq Mod(F'_{i+1})$.

$$\begin{aligned}\bigcup_{j=0}^{i+1} R_j &= \bigcup_{j=0}^{i} R_j \cup R_{i+1} \\ &= Mod(F'_i) \cup R_{i+1} \\ &= Mod(F'_i) \cup (\mathscr{U}_{i+1} \cap Mod(\mathcal{K}_{i+1})) \\ &= Mod(\mathcal{K}_i) \cup (\mathscr{U}_{i+1} \cap Mod(\mathcal{K}_{i+1})) \\ &= (Mod(\mathcal{K}_i) \cup \mathscr{U}_{i+1}) \cap (Mod(\mathcal{K}_i) \cup Mod(\mathcal{K}_{i+1})) \\ &= (Mod(\mathcal{K}_i) \cup \mathscr{U}_{i+1}) \cap Mod(\mathcal{K}_{i+1}) \\ &\subseteq Mod(\mathcal{K}_{i+1}) \\ &= Mod(F'_{i+1})\end{aligned}$$

Next, we show that $Mod(F'_{i+1}) \subseteq \bigcup_{j=0}^{i+1} R_j$.

Suppose for the sake of contradiction that $Mod(F'_{i+1}) \nsubseteq \bigcup_{j=0}^{i+1} R_j$.

$$\begin{aligned}Mod(F'_{i+1}) \nsubseteq \bigcup_{j=0}^{i+1} R_j &\Leftrightarrow Mod(\mathcal{K}'_{i+1}) \nsubseteq \bigcup_{j=0}^{i+1} R_j \\ &\Leftrightarrow \exists v \in Mod(\mathcal{K}'_{i+1}) \text{ s.t. } v \notin \bigcup_{j=0}^{i+1} R_j\end{aligned}$$

Now,

$$\begin{aligned} v \notin \bigcup_{j=0}^{i+1} R_j &\Leftrightarrow v \notin R_j, \forall j \leq i+1 \\ &\Rightarrow v \notin R_{i+1} = \mathscr{U}_{i+1} \cap Mod(\mathcal{K}_{i+1}) \\ &\Rightarrow v \notin Mod(\mathcal{K}_{i+1}) = Mod(\mathcal{K}'_{i+1}) \end{aligned}$$

Thus, we have that $Mod(F'_{i+1}) \subseteq \bigcup_{j=0}^{i+1} R_j$ and consequently, $Mod(F'_{i+1}) = \bigcup_{j=0}^{i+1} R_j$.

Now, if $\mathcal{K}'_{i+1} = \mathcal{K}'_i$, then we have that `CumulativeFormulaRank` terminates. We must now show that `ModelRank` terminates at the same index $(i+1)$.

$$\begin{aligned} R_{i+1} :&= \mathscr{U}_{i+1} \cap Mod(\mathcal{K}_{i+1}) \\ &= \mathscr{U}_{i+1} \cap Mod(\mathcal{K}_i) \\ &= (\mathscr{U}_i \setminus R_i) \cap Mod(\mathcal{K}_i) \\ &= (\mathscr{U}_i \cap Mod(\mathcal{K}_i)) \setminus (R_i \cap Mod(\mathcal{K}_i)) \\ &= R_i \setminus R_i \\ &= \emptyset \end{aligned}$$

Thus `ModelRank` terminates at the same index.

D `LexicographicModelRank` Proofs

Proposition 6. *The* `LexicographicModelRank` *algorithm terminates.*

Proof. The outermost while loop executes exactly n times and should not affect termination. Therefore, termination will depend entirely on whether the inner while loop terminates for each value of i.

For any $i < n$:
Since $\{L_{ij} \mid 0 \leq j \leq \#\mathcal{K}\} \setminus \{\emptyset\}$ partitions $\mathscr{U}_{i0}$, $\bigcup_{j=0}^{\#\mathcal{K}} L_{ij} = \mathscr{U}_{i0} = R_i^{RC}$.

Now, the algorithm recursively defines $\mathscr{U}_{ij}$ as $\mathscr{U}_{i(j-1)} \setminus L_{i(j-1)}$, resulting in the following derivation:

$$\mathscr{U}_{ij} = \mathscr{U}_{i0} \setminus L_{i0} \setminus ... \setminus L_{i(j-1)}$$
$$\implies \mathscr{U}_{ij} = \mathscr{U}_{i0} \setminus \bigcup_{k=0}^{j-1} L_{ik}$$
$$\implies \mathscr{U}_{ij} = R_i^{RC} \setminus \bigcup_{k=0}^{j-1} L_{ik}$$

But for $j = \#\mathcal{K}+1$, we have $\bigcup_{k=0}^{\#\mathcal{K}} L_{ik} = \mathscr{U}_{i0} = R_i^{RC}$, since the L_{ik}'s partition the rank.

Thus, $\mathscr{U}_{ij} = R_i^{RC} \setminus R_i^{RC} = \emptyset$, the required condition for termination of the inner loop.

Therefore, we have that the innermost loop will terminate after at most $\#\mathcal{K} + 1$ iterations, for each value of i, and hence the algorithm terminates.

Proposition 7. *The* `LexicographicModelRank` *algorithm produces the lexicographic [4] ranked model of $\mathcal{K}$.*

Proof. Suppose `LexicographicModelRank` produces
$\mathcal{R}^* = (R_0, ..., R_{n-1}, R_\infty)$.

We will prove the above in two parts:

1. $\mathcal{R}^*$ is a ranked interpretation:
   We show that all worlds are assigned a unique rank, and that there are no empty ranks in the model.

   We have that $\mathcal{R}_{\mathcal{K}}^{RC} = (R_0^{RC}, ..., R_{n-1}^{RC}, R_\infty^{RC})$ produced by `ModelRank` is a ranked interpretation.
   Consider $u \in R_i^{RC}$:
   There is some j such that $u \in L_{ij}$, since $\bigcup_{k=0}^{\#\mathcal{K}} L_{ik} = R_i^{RC}$.
   Since $L_{ij} \neq \emptyset$, there is some k such that $R_k = L_{ij}$ and hence $\mathcal{R}^*(u) = k$.
   This rank is unique since $\nexists L_{i'j'} : u \in L_{i'j'}, i' \neq i$ *or* $j' \neq j$.

This follows from the fact that the rational closure ranks partition $\mathscr{U}$ and each R_i^{RC} is partitioned by the L_{ij}'s (ignoring potentially empty L_{ij}'s). And so, $\nexists k' : \mathcal{R}^*(u) = k' \textit{ and } k' \neq k$, since $R_k = L_{ij}$.

Consider R_i for some i:
$\exists j, k : R_i = L_{jk} \textit{ and } L_{jk} \neq \emptyset$, by construction, and such L_{jk}'s are placed consecutively.
Therefore, there cannot be an empty rank in the interpretation, which is sufficient in satisfying the required convexity property of ranked interpretations.

2. $\mathcal{R}^*$ conforms to the lexicographic ordering [4] defined on $\mathcal{K}$:
We consider the 3 cases in the defined ordering: $m \prec_{LC}^{\mathcal{K}} n$ if and only if $\mathcal{R}_{RC}^{\mathcal{K}}(n) = \infty$, or $\mathcal{R}_{RC}^{\mathcal{K}}(m) < \mathcal{R}_{RC}^{\mathcal{K}}(n)$, or $\mathcal{R}_{RC}^{\mathcal{K}}(m) = \mathcal{R}_{RC}^{\mathcal{K}}(n)$ and m satisfies more formulas than n in $\mathcal{K}$.

Consider arbitrary $u, v \in \mathscr{U}$:
(a) $\mathcal{R}_{RC}^{\mathcal{K}}(v) = \infty$:
Since $R_\infty = R_\infty^{RC}$, $\mathcal{R}^*(v) = \infty$, and hence $u \prec_{\mathcal{R}^*} v$.

(b) $\mathcal{R}_{RC}^{\mathcal{K}}(u) < \mathcal{R}_{RC}^{\mathcal{K}}(v)$:
Then $u \in L_{ij}$ and $v \in L_{kl}$ for some i, j, k, l such that $i < j$. Since $R_m = L_{ij}$ and $R_n = L_{kl}$ for some $m < n$, we have that $R^*(u) < R^*(v)$ and therefore than $u \prec_{\mathcal{R}^*} v$.

(c) $\mathcal{R}_{RC}^{\mathcal{K}}(u) = \mathcal{R}_{RC}^{\mathcal{K}}(v)$ and u satisfies more formulas than v in $\mathcal{K}$:
Let $i = \mathcal{R}_{RC}^{\mathcal{K}}(u) = \mathcal{R}_{RC}^{\mathcal{K}}(v)$. Then, $u \in L_{ij}$ and $v \in L_{ik}$ for some $j < k$, since u satisfies more formulas and hence violates fewer formulas than v in $\mathcal{K}$. Since $R_m = L_{ij}$ and $R_n = L_{ik}$ with $j < k$, we have $m < n$, and hence that $R^*(u) < R^*(v)$ and $u \prec_{\mathcal{R}^*} v$.

We now have that $\prec_{\mathcal{R}^*}$ satisfies all the properties of the lexicographic closure modular ordering, and since it is a ranked interpretation, it must be the unique ranked interpretation obeying such an ordering. From [4], we know that the ranked interpretation corresponding to lexicographic closure is a model of $\mathcal{K}$, and hence $\mathcal{R}^*$ is the lexicographic ranked model of $\mathcal{K}$, as defined by the ordering in [4].

E `LexicographicFormulaRank` Proofs

Proposition 8. *For each rank L'_k in the output of the* ***`LexicographicFormulaRank`*** *algorithm,* $\mathrm{Mod}(L'_k) = L_k$ *where L_k is the corresponding rank in the output of the* ***`LexicographicModelRank`*** *algorithm, with both algorithms returning the same number of ranks.*

Proof. We will first show, inductively, that for each refined rank L'_{ij} in the `LexicographicFormulaRank` algorithm, for any arbitrary i, is such that $Mod(L'_{ij}) =$

L_{ij} where L_{ij} is defined in the `LexicographicModelRank` algorithm, and similarly, that $Mod(\mathscr{U}'_{ij}) = \mathscr{U}_{ij}$. We also show that $\mathscr{U}'_{ij}$ is defined if and only if $\mathscr{U}_{ij}$ is defined.

We first note that $Mod(L'_\infty) = Mod(F^{RC}_\infty) = R^{RC}_\infty = L_\infty$ (we explicitly assign the infinite rank in both algorithms, ensuring correspondence).

Let $i < n$ be any finite rank in any rational closure model with $n-1$ finite ranks.

Base Case:

1. $Mod(\mathscr{U}'_{i0}) = Mod(F^{RC}_i) = R^{RC}_i = \mathscr{U}_{i0}$
2. $\mathscr{U}_{i0} \neq \emptyset$, since the rational closure ranks are non-empty, therefore $\mathscr{U}_{i1}$ and L_{i0} will be defined. Similarly, $\mathscr{U}'_{i0} \not\models \bot$, since $\mathscr{U}'_{i0} \not\models \bot \iff Mod(\mathscr{U}'_{i0}) = \mathscr{U}_{i0} \neq \emptyset$. Therefore, $\mathscr{U}'_{i1}$ and L'_{i0} will be defined.
3. $$\begin{aligned} Mod(L'_{i0}) &= Mod\left(F^{RC}_i \wedge \left(\bigvee_{S\in\{T\subseteq\overrightarrow{\mathcal{K}} \mid \#T=\#\overrightarrow{\mathcal{K}}-0\}} \bigwedge_{s\in S} s\right)\right) \\ &= Mod(F^{RC}_i) \cap \bigcup_{S\in\{T\subseteq\overrightarrow{\mathcal{K}} \mid \#T=\#\overrightarrow{\mathcal{K}}\}} Mod(S) \\ &= R^{RC}_i \cap Mod(\overrightarrow{\mathcal{K}}) (\text{the only subset of size } \#\overrightarrow{\mathcal{K}} \text{ is } \overrightarrow{\mathcal{K}}) \\ &= \mathscr{U}_{i0} \cap Mod(\overrightarrow{\mathcal{K}}) \\ &= \{u \in \mathscr{U}_{i0} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \not\Vdash k\} = 0\} \\ &= L_{i0} \end{aligned}$$

Inductive Step:

Assume for some j such that L_{ij}, L'_{ij} and $\mathscr{U}_{ij}, \mathscr{U}'_{ij}$ are defined, that $L_{ij} = Mod(L'_{ij})$ and $\mathscr{U}_{ij} = Mod(\mathscr{U}'_{ij}) \neq \emptyset$.

1. $$\begin{aligned} Mod(\mathscr{U}'_{i(j+1)}) &= Mod(\mathscr{U}'_{ij} \wedge \neg L'_{ij}) \\ &= Mod(\mathscr{U}'_{ij}) \cap \overline{Mod(L'_{ij})} \\ &= \mathscr{U}_{ij} \setminus L_{ij} \\ &= \mathscr{U}_{i(j+1)} \end{aligned}$$
2. Now,
$$\begin{aligned} \mathscr{U}_{i(j+1)} = \emptyset &\iff Mod(\mathscr{U}'_{i(j+1)}) = \emptyset \\ &\iff \mathscr{U}'_{i(j+1)} \models \bot \end{aligned}$$

Therefore,

$$\begin{aligned} L_{i(j+1)} \text{ is defined } &\iff \mathscr{U}_{i(j+1)} \neq \emptyset \\ &\iff \mathscr{U}'_{i(j+1)} \not\models \bot \\ &\iff L'_{i(j+1)} \text{ is defined .} \end{aligned}$$

3. If $\mathscr{U}_{i(j+1)} = \emptyset$, we are done (both $L_{i(j+1)}$ and $L'_{i(j+1)}$ will not be defined, with L_{ij}, L'_{ij} the last defined ranks for the refinement of rational closure rank i.
Else, $\mathscr{U}_{i(j+1)} \neq \emptyset$ and so $L_{i(j+1)}, L'_{i(j+1)}$ are defined.

$$\begin{aligned} Mod(L'_{i(j+1)}) &= Mod\left(\mathscr{U}'_{i(j+1)} \wedge \left(\bigvee_{S \in \{T \subseteq \overrightarrow{\mathcal{K}} \mid \#T = \#\overrightarrow{\mathcal{K}} - (j+1)\}} \bigwedge_{s \in S} s\right)\right) \\ &= Mod\left(\mathscr{U}'_{ij} \wedge \neg L'_{ij} \wedge \left(\bigvee_{S \in \{T \subseteq \overrightarrow{\mathcal{K}} \mid \#T = \#\overrightarrow{\mathcal{K}} - (j+1)\}} \bigwedge_{s \in S} s\right)\right) \\ &= Mod(\mathscr{U}'_{ij}) \cap \overline{Mod(L'_{ij})} \cap \{u \in \mathscr{U} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \Vdash k\} \geq \#\overrightarrow{\mathcal{K}} - (j+1)\} \\ &= \mathscr{U}_{ij} \cap \mathscr{U} \setminus L_{ij} \cap \{u \in \mathscr{U} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \nVdash k\} \leq j+1\} \\ &= \mathscr{U}_{ij} \setminus L_{ij} \cap \{u \in \mathscr{U} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \nVdash k\} \leq j+1\} \\ &= \mathscr{U}_{i(j+1)} \cap \{u \in \mathscr{U} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \nVdash k\} \leq j+1\} \\ &= \{u \in \mathscr{U}_{i(j+1)} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \nVdash k\} \leq j+1\} \\ &= \{u \in \mathscr{U}_{i(j+1)} \mid \#\{k \in \overrightarrow{\mathcal{K}} \mid u \nVdash k\} = j+1\} \\ &= L_{i(j+1)} \end{aligned}$$

Thus, by induction, the refined ranks in each algorithm correspond as required.

Using this result, since

$$\begin{aligned} L_k = L_{ij} &\iff L_{ij} \neq \emptyset \\ &\iff L'_{ij} \not\models \bot \\ &\iff L'_k = L'_{ij} \end{aligned}$$

we must have that $\forall k \leq n, Mod(L'_k) = L_k$.